Nonconvex Equilibrium Prices in Prediction Markets

Ashwin K. Seshadri

DECISION ANALYSIS
Vol. 12, No. 1, March 2015, pp. 1–14
ISSN 1545-8490 (print) | ISSN 1545-8504 (online)
http://dx.doi.org/10.1287/deca.2014.0306

Nonconvex Equilibrium Prices in Prediction Markets

Ashwin K. Seshadri
Department of Engineering and Public Policy, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213;
and Indian Institute of Science, Bangalore, Karnataka 560012, India, ashwin@fastmail.fm

Prediction markets are increasingly being used to estimate probabilities of future events, and market equilibrium prices depend on the distribution of subjective probabilities of underlying events. When each contract requires the payment of a dollar if the underlying event were to occur, equilibrium prices are usually used to estimate the mean probabilities of the corresponding events. This paper shows that under certain conditions, market equilibrium prices of such contracts can lie outside the convex hull of potential traders' probability beliefs, and where this occurs, market forecasts can induce stochastically dominated group decisions. We describe examples of where this could occur and generalize these examples to characterize conditions for nonconvex prices. A necessary condition for nonconvex prices is that market risk premia for complementary contracts have opposite signs. Preference functions on the lines of prospect theory have this property.

Keywords: probability forecasting; prediction markets; market risk premium; nonconvex prices; stochastic dominance
History: Received April 13, 2014. Accepted by Editor-in-Chief Rakesh K. Sarin on September 23, 2014, after 2 revisions. Published online in *Articles in Advance* November 20, 2014.

1. Introduction

Prediction markets, also known as idea futures or decision markets, trade contracts whose payoffs depend on the occurrence of mutually exclusive and exhaustive future events. When contracts require the payment of a dollar if the corresponding event were to occur, equilibrium prices for these contracts are interpreted as the market's probabilities for these events. Furthermore, for their ability to improve forecasts of uncertain events, prediction markets have been suggested as mechanisms to improve decisions (Hansen 1999, Arrow et al. 2008).

This paper shows that under certain conditions, equilibrium prices in prediction markets can be outside the convex hull of potential traders' probability beliefs. Where this occurs, prediction market forecasts can induce stochastically dominated group decisions (see §3).

The setting we focus on in this paper is binary prediction markets offering contracts defined on mutually exclusive and exhaustive events E_1 and E_2. Contract C_1 requires payment of a dollar if event E_1 occurs and nothing if it does not, whereas C_2 requires payment of a dollar if E_2 occurs and nothing if it does not. Market equilibrium prices for these two events are denoted as c_1 and c_2, respectively. We set up a model with atomic agents and unique probabilities about these events.

Prior studies of equilibrium prices in prediction markets (Wolfers and Zitzewitz 2004, Gjerstad 2005, Manski 2006) have characterized traders' preferences as convex. Observed preferences often do not behave this way. Prospect theory preferences in market settings (Kahneman and Tversky 1979) are suggested by observations of individual betting choices (Andrikogiannopoulou 2011) and the long-shot bias in betting prices (Snowberg and Wolfers 2010). Prospect theory (Kahneman and Tversky 1979) is characterized by traders maximizing a weighted value function, usually defined along a single monetary dimension. Outcome values are evaluated relative to a reference point. Losses have larger effects than equivalent gains, and the sensitivity of the value function diminishes with distance from the reference point. Decision weights for risky events increase in event probabilities but generally differ from them. In binary prediction markets, these weights imply overweighting small probabilities and underweighting large ones when compared with what expected utility maximizers would do. These aspects of the prospect theory model account for experimental evidence on individual choice (Kahneman and Tversky 1979, Tversky and Kahneman 1992) and are supported by evidence from psychology (Helson 1964, Galanter and Pliner 1974, Rabin and Thaler 2001). There are also models of utility maximization in which the utility function contains a local inflexion point (Markowitz 1952), describing different attitudes toward risk when faced with potential gains and losses.

We examine what effects such types of preferences can have. The model describing individual choice is based on cumulative prospect theory (CPT) (Tversky and Kahneman 1992). CPT avoids, at the level of individual decisions, the stochastic dominance arising from nonlinearity of probability weights in the original formulation of prospect theory (Kahneman and Tversky 1979).

2. Individual Preferences and Portfolio Choice in a Binary Prediction Market

Consider a collection of potential traders indexed by i. Their trading decisions depend on the market price, their individual preferences, and their probability beliefs about these events. Potential trader i believes the probability of event E_1 to be p_{1i} and that of complementary event E_2 to be p_{2i} . These probabilities have the relation $p_{1i}+p_{2i}=1$ for every i, so traders are not liable to a Dutch book (de Finetti et al. 1990). We assume that traders each have preferences over lotteries that can be characterized by maximization of a preference function.

In our idealized market, potential trader i faces the choice of not participating or of purchasing and holding n_{1i} and n_{2i} units of contracts C_1 and C_2, respectively, where n_{1i} and n_{2i} are real numbers. These are obtained as solutions to the constrained optimization problem characterizing the trader's choice, described below.

Both contracts require the payment of a dollar if and when the corresponding event occurs. After purchasing n_{1i} and n_{2i} contracts, where $(n_{1i}, n_{2i}) \in \mathbb{R}_+^2$, trader i faces the lottery $\{y_{1i}, p_{1i}; y_{2i}, p_{2i}\}$, where $y_{1i}=-n_{1i}c_1-n_{2i}c_2+n_{1i}$ and $y_{2i}=-n_{1i}c_1-n_{2i}c_2+n_{2i}$ are the payouts, net of purchasing costs. For simplicity, we neglect time discounting.

Acts correspond to the choice of (n_{1i}, n_{2i}). Without loss of generality, the value of the preference function when potential trader i does not participate in the market is 0; i.e., $V_i(0,0)=0$. Potential trader i will participate in the market if (n_{1i}, n_{2i}) can be chosen to make $V_i(n_{1i}, n_{2i})>0$, subject to the trader's budget constraint. This budget constraint is denoted by b_i.

We assume there are no openings for risk-free profit to be made in the market, i.e., for arbitrage. Then, $c_1+c_2=1$. Otherwise, for example, if $c_1+c_2<1$, risk-free profit could be made by buying an equal number of units of both contracts to be held until the payoffs are implemented. We have implicitly assumed that there are no commissions in the market.[1]

Under the no-arbitrage condition, the payouts are $y_{1i}=(n_{1i}-n_{2i})(1-c_1)$ and $y_{2i}=-(n_{1i}-n_{2i})c_1$. Because $c_1 \le 1$, these payouts are of opposite signs except when c_1 is equal to 1 or 0. Therefore, the payout in the market is a mixed lottery; i.e., one event will produce a gain and the other a loss.

According to the model of CPT (Tversky and Kahneman 1992), the overall value is the sum of positive and negative values that separately account for positive and negative monetary outcomes, respectively. Although decision weights generally can differ between these two types of outcomes, we make the simplifying assumption that positive and negative outcomes have the same decision weights that depend only on the probabilities of the events. Because the payout in the market is a mixed lottery, the overall value can be described as

$$\begin{aligned} V_i(n_{1i}, n_{2i}) &= w_i(p_{1i})v_{1i}(y_{1i}(n_{1i}, n_{2i})) \\ &\quad + w_i(p_{2i})v_{2i}(y_{2i}(n_{1i}, n_{2i})), \end{aligned} \tag{1}$$

where w_i is the probability weighting function and v_i is the value function. We assume the standard form of the value function described by prospect theory, $v_i(y)=y^{a_i}$ for $y \ge 0$ and $v_i(y)=-\Gamma_i(-y)^{a_i}$ for $y<0$, with $a_i \le 1$ and $\Gamma_i \ge 1$. Strict loss aversion corresponds to $\Gamma_i>1$, whereas diminishing sensitivity to increased changes corresponds to $a_i<1$.

The choice facing potential trader i can be cast as the following maximization problem:

$$\max_{n_{1i}, n_{2i}} V_i(n_{1i}, n_{2i}), \tag{2}$$

subject to the trader's budget constraint $n_{1i}c_1 + n_{2i}c_2 \le b_i$.

[1] In reality, $c_1+c_2>1$ even in a market with no arbitrage, to allow commissions to be paid.

The function $v_i(y)$ is increasing in y for all traders i. Furthermore, the probability weighting function $w_i(p)$ increases with probability p. We call such preferences where the value function and probability weighting function are increasing in their respective arguments *increasing preferences*. Increasing preferences consider only those preference functions in which higher monetary amounts are preferred and higher probabilities corresponding to identical positive outcomes are preferred. The preferences of CPT are a special case of increasing preferences. The corresponding space of all functions describing increasing preferences is denoted by $\mathcal{F}_{\mathcal{J}}$.

We adopt CPT (Tversky and Kahneman 1992) and assume that the probability weighting function for gains is identical to that for losses. If trader i is long on contract C_1, i.e., $y_{1i} > 0$ and $y_{2i} < 0$, then the overall value $V_i(n_{1i}, n_{2i})$ is given by

$$V_i(n_{1i}, n_{2i}) = w_i(p_{1i})(-n_{1i}c_1 - n_{2i}c_2 + n_{1i})^{a_i} - \Gamma_i w_i(p_{2i})(n_{1i}c_1 + n_{2i}c_2 - n_{2i})^{a_i}, \quad (3)$$

whereas if the trader is long on contract C_2, with $y_{2i} > 0$ and $y_{1i} < 0$, then

$$V_i(n_{1i}, n_{2i}) = -\Gamma_i w_i(p_{1i})(n_{1i}c_1 + n_{2i}c_2 - n_{1i})^{a_i} + w_i(p_{2i})(-n_{1i}c_1 - n_{2i}c_2 + n_{2i})^{a_i}. \quad (4)$$

A consequence of the no-arbitrage condition is that potential traders might choose to hold one contract or the other, but not both.[2] This applies for arbitrary preference functions in CPT and requires only the absence of arbitrage openings in the market. Diversification is not advantageous to traders in a binary prediction market because the two complementary contracts present different expected returns.

Hence we can describe the choice facing a potential trader as involving the selection of n_{1i} to maximize

$$V_i(n_{1i}, 0) = w_i(p_{1i})(-n_{1i}c_1 + n_{1i})^{a_i} - \Gamma_i w_i(p_{2i})(n_{1i}c_1)^{a_i} \quad (5)$$

or n_{2i} to maximize

$$V_i(0, n_{2i}) = -\Gamma_i w_i(p_{1i})(n_{2i}c_2)^{a_i} + w_i(p_{2i})(-n_{2i}c_2 + n_{2i})^{a_i}, \quad (6)$$

subject to the trader's budget constraint. Differentiating Equation (5) with respect to n_{1i} yields

$$\frac{\partial V_i(n_{1i}, 0)}{\partial n_{1i}} = a_i n_{1i}^{a_i - 1}(w_i(p_{1i})(1-c_1)^{a_i} - \Gamma_i w_i(p_{2i})c_1^{a_i}). \quad (7)$$

Therefore, the maximizing solution for potential trader i would be to spend his entire budget holding C_1 if

$$\frac{w_i(p_{1i})}{w_i(1-p_{1i})} > \Gamma_i\left(\frac{c_1}{1-c_1}\right)^{a_i}, \quad (8)$$

whereas from Equation (6), the maximizing solution would be to spend his budget holding C_2 if

$$\frac{w_i(1-p_{1i})}{w_i(p_{1i})} > \Gamma_i\left(\frac{1-c_1}{c_1}\right)^{a_i}. \quad (9)$$

In the absence of strict loss aversion, i.e., with $\Gamma_i = 1$, either one of the two inequalities in Equations (8) and (9) must be satisfied. In this case the potential trader would spend his entire trading budget holding one of the contracts. However, if $\Gamma_i > 1$, then it is possible that the trader chooses not to enter the market.

3. Nonconvex Market Equilibrium and Stochastic Dominance

In this and the following sections, we make the restriction that potential traders in the market all have increasing preferences; i.e., $V_i \in \mathcal{F}_{\mathcal{J}}$. As described earlier, this imposes weak conditions on the preference function so that individuals prefer higher monetary outcomes to lower ones, if both are certain, and higher probabilities to lower ones, if both are associated with identical positive outcomes. The preferences of CPT

[2] If, for example, trader i is long on C_1, the sensitivity of overall value to increasing holdings of C_1 is

$$\frac{\partial V_i(n_{1i}, n_{2i})}{\partial n_{1i}} = a_i(1-c_1)w_i(p_{1i})(-n_{1i}c_1 - n_{2i}c_2 + n_{1i})^{a_i-1} - a_i c_1 \Gamma_i w_i(p_{2i})(n_{1i}c_1 + n_{2i}c_2 - n_{2i})^{a_i-1},$$

whereas the corresponding sensitivity to increasing holdings of C_2 is

$$\frac{\partial V_i(n_{1i}, n_{2i})}{\partial n_{2i}} = -a_i c_2 w_i(p_{1i})(-n_{1i}c_1 - n_{2i}c_2 + n_{1i})^{a_i-1} + a_i(1-c_2)\Gamma_i w_i(p_{2i})(n_{1i}c_1 + n_{2i}c_2 - n_{2i})^{a_i-1}.$$

Under the no-arbitrage condition, $c_1 + c_2 = 1$, and therefore,

$$\frac{\partial V_i(n_{1i}, n_{2i})}{\partial n_{1i}} = -\frac{\partial V_i(n_{1i}, n_{2i})}{\partial n_{2i}};$$

i.e., it is equally advantageous to either increase holdings of one of the contracts or decrease holdings of the other. If $\partial V_i(n_{1i}, n_{2i})/\partial n_{1i} > 0$, the trader would end up holding only C_1. On the contrary, if $\partial V_i(n_{1i}, n_{2i})/\partial n_{2i} > 0$, the trader would hold only C_2.

satisfy this condition. Before describing the consequences of using nonconvex equilibria for making decisions, we must define the notions of *inducing an act* and *stochastic dominance*.

A prediction market is said to induce an act A_d if, according to the market forecast, each individual i with a preference function belonging to the space of increasing preferences, i.e., $V_i \in \mathscr{F}_{\mathscr{I}}$, prefers act A_d compared with all other available acts. This preference for act A_d must hold for any preference function belong to $\mathscr{F}_{\mathscr{I}}$.

We now define stochastic dominance. An act A_d is said to be stochastically dominated for the set of increasing preferences $\mathscr{F}_{\mathscr{I}}$ if there exists another act A_s with $s \neq d$ such that A_s is preferred to A_d for all preference functions belonging to $\mathscr{F}_{\mathscr{I}}$. Stochastic dominance characterizes changes in risk that have unambiguous effects on preferences (Gollier 2001); here, the definition is restricted to preference functions contained within $\mathscr{F}_{\mathscr{I}}$.

The following definition is central to this section so is made explicit.

Definition 1 (Convex Hull). The convex hull of potential traders' beliefs in a binary prediction market is the minimal convex set in $\mathbb{R}^2_+$ generated by all linear combinations of the form

$$\alpha_i \vec{p}_i, \tag{10}$$

where $0 < \alpha_i < 1$, $\sum \alpha_i = 1$, and $\vec{p}_i \equiv \{p_{1i}; p_{2i}\}$ is the set of the duplex of probabilities about complementary events E_1 and E_2, respectively, across all potential traders. Equivalently, the convex hull can be characterized in terms of the intervals

$$\left[\min_i p_{ji}, \max_i p_{ji}\right] \tag{11}$$

for $j = 1, 2$, which denotes the event in question. In summary, the convex hull is the result of all linear combinations of probabilities among traders, where the weights are nonnegative and sum to 1; this is characterized by intervals bounded by the minimum and maximum of the probabilities for E_1 and E_2 among all traders.

We are now ready to introduce the main result of this section, which describes how prices outside the convex hull of probabilities can induce acts that are stochastically dominated. These latter notions were defined in the beginning of this section.

Proposition 1. *Consider a binary prediction market with contracts based on mutually exclusives and exhaustive events E_1 and E_2. Contract C_1 requires the payment of a dollar if E_1 occurs, and C_2 requires the payment of a dollar if E_2 occurs. Potential traders have preference functions belonging to the function space $\mathscr{F}_{\mathscr{I}}$; i.e., they have increasing preferences. Then if market equilibrium price c_1 for contract C_1 lies outside the convex hull of probability beliefs about E_1, such an equilibrium can induce particular acts from within the set of acts facing all traders that are stochastically dominated for the set of increasing preferences.*

Proof. Define act A_j as "receive \$1 if event E_j occurs upon a payment now of \$$X$," where $X < 1$ and is known. Then $A_k \succ A_l$ (i.e., A_k is preferred to A_l) by potential trader i if and only if $p_i(E_k) > p_i(E_l)$; i.e., i prefers act A_k to A_l if and only if his belief is that E_k is more probable than E_l. Consider the set of events $\{E_1, E_f\}$ and corresponding acts $\{A_1, A_f\}$ that are available to the group of potential traders, where (i) $p(E_1)$ is unknown so that $p_i(E_1)$ differs between potential traders, (ii) $p(E_f)$ is known and $p_i(E_f) = p(E_f)$ for all potential traders i, and (iii) $p(E_f) > \max p_i(E_1)$. Then $A_f \succ A_1$ for all i if $V_i \in \mathscr{F}_{\mathscr{I}}$; i.e., A_1 is stochastically dominated for the set of increasing preferences.

If the market equilibrium price for C_1 is larger than the known probability of E_f, i.e., $c_1 > p(E_f)$, then the estimate of $p(E_1)$ is larger than the known $p(E_f)$, so the market induces act A_1. From Definition 1, in this condition, c_1 is outside the convex hull of $\{p_{1i}\}$. This example proves the proposition that market equilibrium price outside the convex hull of traders' probability beliefs about E_1 can induce particular acts from within the set of acts facing all traders that are stochastically dominated in the space of all increasing preferences.

Example. This example illustrates how a market with an equilibrium price outside the convex hull of probability beliefs can induce acts that are stochastically dominated. Consider a firm trying to decide between two investments; we call these "Risky" and "Uncertain." Risky has a known probability of success, $p_R = 0.6$. The success of investment Uncertain depends on the occurrence of a future event S whose probability is unknown; let us call this probability p_S. Each investment entails an up-front cost $D = 5$ million dollars and, if successful, a constant return $R = 10$ million dollars.

Now imagine a decision maker seeking to estimate the probability p_S by creating an internal prediction market to trade contracts defined on the occurrence and nonoccurrence of S. That is, contract C_1 requires the payment of a dollar if S occurs and contract C_2 requires the payment of a dollar if an event other than S occurs. Probability beliefs about S among potential traders have a convex hull $[p_{S,\min}, p_{S,\max}]$.

Suppose that $p_R = 0.6 > p_{S,\max}$; i.e., Risky has a known probability of success that is larger than the individual probability beliefs about the success of Uncertain. Then because potential traders all have increasing preferences, they all prefer the lottery $\{-D, 1-p_R; R-D, p_R\}$ to the lottery $\{-D, 1-p_S; R-D, p_S\}$. Therefore, Uncertain is stochastically dominated by Risky.

To illustrate the effects of prices outside the convex hull of probabilities, let us furthermore imagine the case that the equilibrium price for C_1 is $c_1 = 0.65 > p_R = 0.6$. Then the market forecast for the probability of S is larger than the known probability p_R. Therefore such a market forecast induces the stochastically dominated act of Uncertain.

Such a situation where a market can induce stochastically dominated acts requires that $c_1 > p_R > p_{S,\max}$. In other words, selection of the stochastically dominated act requires that the market forecast for the probability of S, i.e., c_1, be outside the convex hull of probability beliefs about event S.

Proposition 2 (Converse). *For the binary prediction market considered in Proposition 1 in which potential traders all have increasing preferences, if the market equilibrium price c_1 lies within the convex hull of probability beliefs about E_1, such an equilibrium cannot induce acts facing all traders that are stochastically dominated for the set of increasing preferences.*

Proof. For A_1 to be stochastically dominated for the set of increasing preferences, $A_f \succ A_1$ for all i if $V_i \in \mathcal{F}_{\mathcal{I}}$. Given the definition of the acts A_j (i.e., "receive \$1 if event E_j occurs upon a payment now of \$$X$"), stochastic dominance of A_1 by A_f requires $p(E_f) > p_i(E_1)$ for all i. Therefore, $p(E_f) > \max p_i(E_1)$. By assumption, the market price lies in the convex hull of probability beliefs, so from the definition of the convex hull, $c_1 < \max p_i(E_1)$. Thus the market equilibrium induces act A_f and not the stochastically dominated act A_1; this proves the proposition.

Nonconvex prices can lead to a selection of alternatives that are stochastically dominated, as the example presented earlier in this section shows. Because probability forecasting is meant to improve decisions before uncertainty is resolved, selection of stochastically dominated alternatives is incoherent ex ante, i.e., in relation to beliefs of market participants before uncertainties are resolved. With nonconvex market equilibrium prices, the market forecast is unfaithful to traders' beliefs, and there can exist ex ante decisions involving the uncertain event where stochastically dominated alternatives are chosen.

4. Model of Market Equilibrium

This section introduces the model describing the equilibrium in our market. Individual traders can choose to hold one or other complementary contract, but not both. As shown in §2, this occurs for arbitrary preferences in CPT and only requires no arbitrage. Because each contract in the market requires the payment of a dollar if the corresponding event were to occur, no arbitrage involves the condition on prices that $c_1 + c_2 = 1$. This assumes that there are no commissions. For simplicity, discounting in time is neglected.

In our market model, trader i entering the market with budget b_i is given an endowment of b_i "unit-contracts" consisting of one unit each of C_1 and C_2. This governs the market supply for each contract. Each contract can be exchanged in the market at the market price, so the model represents an exchange market in which the initial endowment of unit-contracts governs the supply of each of these contracts.[3]

Below we present the key expression for the equilibrium condition in the market for C_1, used to calculate equilibrium price. The left side describes supply of contract C_1 and the right side the demand. Market equilibrium can be written, following Manski (2006), as

$$\sum_i b_i I_{T_{1i} \cup T_{2i}} = \frac{1}{c_1} \sum_i b_i I_{T_{1i}}, \tag{12}$$

[3] This supply is the same for both contracts and the market equilibrium condition, where demand equals supply:

$$\sum_i n_{1i} = \sum_i n_{2i};$$

that is, the total demand for both contracts is the same.

where event T_{1i} is defined as potential trader i choosing to hold C_1, event T_{2i} is defined as i choosing to hold C_2, and b_i is the trading budget of i. Note that T_{1i} and T_{2i} are events defined on the sample space (n_{1i}, n_{2i}) of outcomes for trader i, with T_{1i} occurring if $n_{1i} > 0$ and T_{2i} occurring if $n_{2i} > 0$. The indicator function of compound event $T_{1i} \cup T_{2i}$ is denoted by $I_{T_{1i} \cup T_{2i}}$; this equals 1 if $T_{1i} \cup T_{2i}$ occurs, i.e., if the potential trader chooses to hold either contract so that $n_{1i} + n_{2i} > 0$, and equals 0 otherwise. Likewise, $I_{T_{1i}}$ is the indicator function of the event T_{1i}, which is equal to 1 if i chooses to hold C_1 so that $n_{1i} > 0$ and equal to 0 otherwise.

Summations in the above expression are over the population of potential traders. Moreover, $I_{T_{1i} \cup T_{2i}}$ is not necessarily equal to 1. For example, in the presence of loss aversion or risk aversion, potential traders might choose to hold neither contract; i.e., some potential traders might not enter the market.

The right side describes demand for the contract because any trader i who chooses to hold this contract can purchase b_i/c_1 units of it. This assumes that any potential trader who chooses to hold C_1 will not hold C_2, and his entire budget is spent holding C_1.[4]

For example, if potential trader k's budget is \$100, and k has chosen to enter the market so that $I_{T_{1k} \cup T_{2k}}$ is equal to 1, then by entering the market, this individual has made available 100 unit-contracts. This increases the supply of C_1 in the exchange market by 100 units. Likewise, the supply of C_2 is increased by the same amount.

An analogous expression can be written for equilibrium in the market for C_2, but this is unnecessary because it would only yield the no-arbitrage condition $c_1 + c_2 = 1$, which we have assumed applies.[5]

Equilibrium in the market for C_1 is determined jointly by the equilibrium condition in Equation (12) and the expressions describing portfolio selection arising from Equation (2).

[4] This is based on the result that traders with preferences following CPT will hold only one contract or the other in a binary prediction market, in the absence of arbitrage openings (see §2).

[5] But if we were to not make this assumption explicitly, the absence of arbitrage openings follows from simultaneous equilibrium in markets for C_1 and C_2.

5. Case Studies to Illustrate Nonconvex Prices

This section describes examples of preference functions leading to nonconvex equilibrium prices in the idealized binary prediction market. Nonconvex prices refer to equilibrium prices outside the convex hull (Definition 1) of corresponding probability beliefs.

5.1. Diminishing Sensitivity to Deviations from an Exogenous Reference Point

Reference-dependent preferences characterize the tendency of people to evaluate changes to wealth rather than make assessments of absolute wealth. Prospect theory (Kahneman and Tversky 1979, Tversky and Kahneman 1992) is a prominent model that includes reference dependence. This model also exhibits diminishing sensitivity as the departure from the reference point increases. This is motivated partly by diminishing perceptions of increasing deviations from the reference point (Kahneman and Tversky 1979). Reference dependence combined with diminishing sensitivity can produce nonconvex prices in binary prediction markets, as the following example shows. The example is confined to isolating this effect, so probabilities of chance events are given weights that are equal to the probabilities, and loss aversion is not considered.

Diminishing sensitivity is represented by the relationship $v_i(y) = y^{a_i}$ for certain outcomes with $y > 0$ and $v_i(y) = -(-y)^{a_i}$ for certain outcomes with $y < 0$, where the exponent has the relation $a_i < 1$. Here, we do not consider effects of probability weighting (i.e., we set $w_i(p) = p$) and ignore loss aversion (i.e., we set $(\Gamma_i = 1)$.

With this simplification, the choice that maximizes the preference function, from Equations (8) and (9), is to spend the entire trading budget holding C_1 if

$$\frac{p_{1i}}{1 - p_{1i}} > \left(\frac{c_1}{1 - c_1}\right)^{a_i}; \tag{13}$$

otherwise, traders should spend their entire budget holding C_2.

To illustrate nonconvex prices, we make two simplifications. First, all traders have homogeneous preferences; i.e., $a_i = a$. Second, across the population of potential traders, the probability distribution for E_1 to occur is continuous and uniformly distributed between $p_{1,\min}$ and $p_{1,\max}$. Thus we can write $p_{1i} = p_{1,\min} + i(p_{1,\max} - p_{1,\min})$. That is, with a uniform distribution of probability beliefs, i (where $0 \leq i \leq 1$) can index potential traders' location on the support $[p_{1,\min}, p_{1,\max}]$ of this distribution.

With both these simplifications, the market equilibrium price for C_1 is obtained from Equation (12) as

$$c_1 = \frac{p_{1,\max} - p_t}{p_{1,\max} - p_{1,\min}}, \tag{14}$$

where p_t is the minimum value of p_i at which C_1 is chosen because the condition in Equation (13) is met. Solving this equation yields $p_t = c_1^a/(c_1^a + (1 - c_1)^a)$. The reader can verify from this equation that if $a = 1$, in the limiting case where diminishing sensitivity is absent, the condition for a potential trader to hold contract C_1 is that $p_{1i} > c_1$. Substituting the expression for p_t into Equation (14) yields

$$\frac{1}{c_1}\left(p_{1,\max} - \frac{c_1^a}{c_1^a + (1 - c_a)^a}\right) = 2W_p, \tag{15}$$

where $2W_p \equiv p_{1,\max} - p_{1,\min}$ is the width of the support of the distribution of probability beliefs.

Proposition 3. *We consider traders that are homogeneous in their preferences, have increasing preferences, and have a market with a uniform distribution of beliefs across potential traders. Furthermore, traders all have the same exogenous reference-dependent preferences, assign probability weights equal to probabilities, and do not exhibit loss aversion. Then, strictly diminishing sensitivity is a necessary condition for nonconvex prices and can lead to nonconvex prices if additional relations between the distribution of beliefs and the common preference function hold. Specifically, nonconvex prices will occur if the maximum probability among traders for one of the events is larger than 1/2 and the support of the distribution of probability beliefs is sufficiently narrow.*

Proof. We show that under certain conditions it is possible for $c_1 > p_{1,\max}$ to occur. From Equation (15), this condition is equivalent to

$$\frac{1}{1 - 2W_p} < p_{1,\max}\left(1 + \left(\frac{1}{p_{1,\max}} - 1\right)^a\right). \tag{16}$$

If $a = 1$, the right side of the above equation is equal to 1, so the condition cannot be satisfied because $0 < W_p < 0.5$. This proves the first part of the proposition, that strictly diminishing sensitivity is necessary for nonconvex prices.

Writing the right side of Equation (16) in the form $f_d(p_{1,\max}, a)$, the partial derivative with respect to a is

$$\frac{\partial f_d}{\partial a} = p_{1,\max}\left(\frac{1}{p_{1,\max}} - 1\right)^a \ln\left(\frac{1}{p_{1,\max}} - 1\right), \tag{17}$$

which is negative if $p_{1,\max} > 1/2$. Under this condition the right side of Equation (16) is larger than 1 if $a < 1$. In this case the inequality in Equation (16) is satisfied if W_p is sufficiently small. This proves the second part.

5.2. Probability Weighting

Probability-dependent weights reflect departures from expected utility maximization and characterize certain models of individual choice including prospect theory. Here, we do not consider effects of loss aversion or diminishing sensitivity of the value function. Therefore, the preference function under uncertainty for potential trader i is obtained by setting $a_i = 1$ and $\Gamma_i = 1$ in Equations (3) and (4) so that these equations reduce to a single equation:

$$\begin{aligned} V_i(n_{1i}, n_{2i}) = {} & w_i(p_{1i})y_{1i}(n_{1i}, n_{2i}) \\ & + w_i(p_{2i})y_{2i}(n_{1i}, n_{2i}). \end{aligned} \tag{18}$$

Parameterizing w_i linearly as $w_i(p) = (1 - m_i)p + m_i d_i$, for p different from 0 and 1, gives a reasonable linear approximation to the probability weighting function estimated for cumulative prospect theory by (Tversky and Kahneman 1992) within probability interval $[0.15, 0.85]$. The constraint on this parameterization is that $0 \leq m_i < 1$ and $d_i > 0$, and consequently, probabilities below d_i are overweighted when $m_i > 0$. Using this parameterization yields the result, from Equations (8) and (9), that potential trader i will spend his entire budget holding units of C_1 if

$$\frac{(1 - m_i)p_{1i} + m_i d_i}{(1 - m_i)(1 - p_{1i}) + m_i d_i} > \frac{c_1}{1 - c_1}. \tag{19}$$

The reader can verify that if $m_i = 0$, i.e., $w_i(p) = p$, the condition for potential trader i to purchase C_1 is that $p_{1i} > c_1$.

We solve Equation (19) together with the market equilibrium condition for contract C_1 under the same restrictions on the market as §5.1. That is, traders all have homogeneous preferences so that $m_i = m$ and $d_i = d$, and there is a uniform distribution of probability beliefs across potential traders. This yields the market equilibrium price

$$c_1 = \frac{(\bar{p}_1 + W_p)(1-m) + md}{1 - m(1-2d) + 2W_p(1-m)} \tag{20}$$

for contract C_1, where $\bar{p}_1 = (p_{1,\min} + p_{1,\max})/2$.

Proposition 4. *We consider traders who are homogeneous in their preferences, have increasing preferences, and have a market with a uniform distribution of beliefs across potential traders. Furthermore, the traders all have CPT preferences and the same probability-dependent weights for uncertain and risky outcomes, described by a quasilinear probability weighting function. There is no loss aversion or diminishing sensitivity of the value function. Then it is necessary for weights to differ from the probabilities for nonconvex prices to occur. Nonconvex prices result if additional relations between the distribution of beliefs and the common preference function hold. Specifically, nonconvex prices result if the maximum probability for one of the contracts among traders is smaller than* $1/2$ *and the support of the distribution of probability beliefs is sufficiently narrow.*

Proof. We show that that under certain conditions it is possible for $c_1 > p_{1,\max}$ to occur. From Equation (20), this condition is equivalent to

$$W_p < \frac{md}{1-m}\frac{1-2p_{1,\max}}{2p_{1,\max}}. \tag{21}$$

If weights are equal to probabilities, i.e., $m = 0$, the right side of Equation (21) is 0, and the inequality therein cannot be satisfied. This proves the first part of the proposition, that weights must differ from probabilities for nonconvex prices to occur.

If weights differ from probabilities such that $m > 0$, the right side is positive where $p_{1,\max} < 1/2$ so that the inequality is satisfied if W_p is sufficiently small. This proves the second part of the proposition.

5.3. Numerical Illustrations

Figure 1 plots a dimensionless measure of the nonconvexity of equilibrium prices in the presence of effects of probability weighting in CPT. This measure is defined as $H \equiv (c_1 - p_{1,\max})/2W_p$. Where this quantity is positive, it can be interpreted as a measure of the distance of equilibrium price from the support of beliefs (normalized by the width of the support of beliefs).

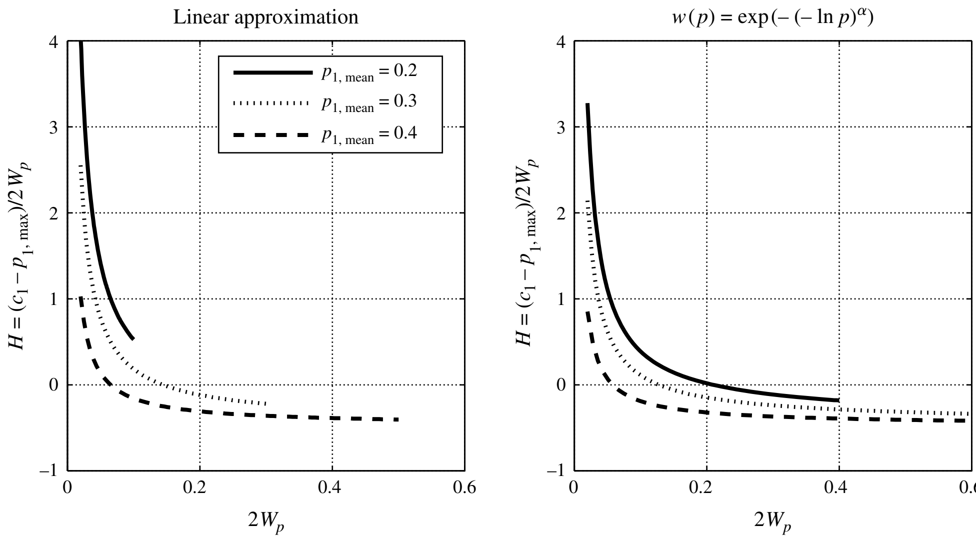
Curves in the left panel use the linear approximation to the weighting function that was discussed previously in this section. Shown are curves for different mean probabilities. Experimental findings on individual choice under risk suggest the form $w(p) = p^\gamma/(p^\gamma + (1-p)^\gamma)^{1/\gamma}$, with the mean value of γ estimated as approximately $\gamma = 0.65$ (Tversky and Kahneman 1992). As discussed previously in this section, we fit a linear model of the form $w_i(p) = (1-m_i)p + m_i d_i$ to the function noted by (Tversky and Kahneman 1992). Linear regression of $w(p)$ versus p, over the range $0.15 \le p \le 0.85$, yields $w(p) = 0.62p + 0.13$, and this model has $R^2 = 0.99$. This fit was the justification for the linear approximation introduced in §5.2. Assuming that traders have homogeneous beliefs, the corresponding parameter estimates are $m = 1 - 0.62 = 0.38$ and $d = 0.13/m = 0.34$. Because the linear approximation applies only for the probability interval $[0.15, 0.85]$, there is an additional constraint on values of W_p. Given the mean probability, W_p cannot be so large as to have minimum probability lower than 0.15. These estimates and constraints are involved in generating the curves in the left panel.

The left panel shows that as the support becomes narrower, H increases. When H is positive, i.e., $c_1 > p_{1,\max}$, it indicates equilibrium prices outside the convex hull of probability beliefs. For narrow support, the value of H can be much larger than 1. In such cases, the equilibrium price is a poor measure of potential traders' probability beliefs.

For smaller mean probability, the value of H is larger. In other words, it becomes easier to obtain prices outside the convex hull of probability beliefs.

The right panel shows the effect of the probability weighting function $w(p) = \exp\{-(-\ln p)^\alpha\}$ with $\alpha = 0.65$, based on the model and estimate suggested by Prelec (1998). Calculations for the curves in the right panel were done numerically, because we cannot obtain an expression for market equilibrium price with this model of the probability weighting function. It should not be surprising that the two panels show similar estimates for the value of H, because the estimate for α in the model of Prelec is also based on experimental findings of Tversky and

Figure 1 Effects of Probability Weighting on the Measure of Nonconvex Equilibrium Prices $H \equiv (c_1 - p_{1,\max})/2W_p$, as the Width $2W_p$ of the Support of the Distribution of Probability Beliefs Is Varied

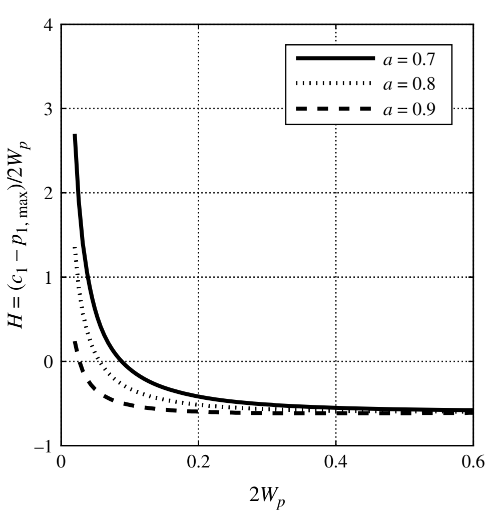


Notes. The left panel shows the effect of the linear approximation to the probability weighting function of (Tversky and Kahneman 1992) for which analytical results were presented in §5.2. The right panel shows numerical estimates based on the functional form $w(p) = \exp\{-(-\ln p)^{\alpha}\}$ with $\alpha = 0.65$ (Prelec 1998).

Kahneman (1992). However, the linear approximation overestimates the value of H for small values of $2W_p$ when compared with the more accurate model suggested by Prelec.

Figure 2 shows the effects of diminishing sensitivity of the value function. Here, the mean probability of event E_1 is fixed at 0.7, and different curves show corresponding results for different values of the exponent a. If the support is narrower, H is larger. Smaller values of the exponent a corresponding to more strongly diminishing sensitivity of the value function lead to larger values of H. Here, too, H can take values larger than 1, if the support is narrow.

Figure 2 Effects of Diminishing Sensitivity of the Value Function on the Measure of Nonconvex Equilibrium Prices $H \equiv (c_1 - p_{1,\max})/2W_p$, as the Width $2W_p$ of the Support of the Distribution of Probability Beliefs Is Varied



6. Generalization

This section generalizes the above results to describe conditions under which nonconvex prices occur in binary prediction markets. The generalization also includes situations where traders might choose neither contract, as can occur in the presence of loss aversion or risk aversion. The characterization is in terms of market risk premium, defined below.

Definition 2 (Market Risk Premium). The market risk premium of potential trader i for contract C_j, $j = 1, 2$ is the excess minimum probability of event E_j for the trader to hold the contract compared to if he had been risk neutral. That is, the market risk premium is $r_{ji} = p_{ji,t} - p_{ji,trn}$, where $p_{ji,t}$ is the minimum probability for potential trader i to hold contract j and $p_{ji,trn}$ is the corresponding value under the hypothetical condition that i is risk neutral.

Lemma 1 (Relation to Market Price). *The market risk premium of potential trader i for contract C_j is equal to $p_{ji,t} - c_j$, i.e., the difference between the minimum probability of E_j at which they would hold C_j and the price c_j.*

Proof. From Definition 2, the market risk premium is $r_{ji} = p_{ji,t} - p_{ji,trn}$. The latter term on the right side is the minimum probability for trader i to hold contract j if he were risk neutral, and it can be obtained from Equation (13) by setting $a_i = 1$ or, equivalently, from Equation (19) by setting $m_i = 0$, both of which yield $p_{ji,trn} = c_j$. This proves the lemma.

The market risk premium is a measure of risk aversion of i's preference function. In general, it is a function of market price of the contract. Therefore, in general, $r_{1i} \neq r_{2i}$—that is, the market risk premia of potential trader i for the two complementary contracts are different.

Lemma 2 (Homogeneous Preferences). *In an idealized market where all traders have the same preferences, the market risk premium for a particular contract is the same among all traders; i.e., $r_{ji} = r_j = p_{j,t} - c_j$, $j = 1, 2$. The quantity r_j is called the market risk premium of the homogeneous market for contract C_j.*

Proof. When traders all have the same preferences, the risk premium is only a characteristic of the common preference function and the particular contract in consideration. Specifically, when the preference function is common for all i, the quantity $p_{ji,t} - c_j$ is independent of i, thus proving the lemma.

This also shows that when traders have the same preferences, the minimum probability of E_j at which C_j is chosen is independent of i.

Using the above definition of the market risk premium, we can obtain insight into equilibrium prices. From the market equilibrium condition in Equation (12) and under the same restrictions as before (homogeneous preferences, uniform distribution of probability beliefs among traders), the market price for C_1 is

$$c_1 = \frac{p_{1,\max} - p_{1,t}}{(p_{1,\max} - p_{1,t}) + (p'_{1,t} - p_{1,\min})}, \tag{22}$$

where $p_{1,t}$ is the minimum value of the probability about E_1 at which C_1 is chosen and $p'_{1,t}$ is the maximum value of the probability about E_1 at which C_2 is chosen. From Lemma 2, $p_{1,t} = c_1 + r_1$, where r_1 is the market risk premium of the homogenous market for C_1. Analogously, $p_{2,t} = c_2 + r_2$. As for $p'_{1,t}$, it corresponds to the minimum value of the probability about E_2 at which C_2 is chosen (i.e., to $p_{2,t}$), and the relation between the two is $p'_{1,t} = 1 - p_{2,t}$. Substituting these relations in the above equation yields

$$c_1 = \frac{p_{1,\max} - (c_1 + r_1)}{2W_p - (r_1 + r_2)}. \tag{23}$$

The denominator describes that portion of the support of probability beliefs corresponding to participation in the market, entailing the selection by traders of either contract. This term also indicates the constraints that sum $r_1 + r_2$ of market risk premia must obey in a functioning market. First,

$$r_1 + r_2 < 2W_p, \tag{24}$$

or else no traders choose either contract. Second,

$$r_1 + r_2 \geq 0 \tag{25}$$

because if $r_1 + r_2 < 0$, then some traders prefer both contracts, which is not possible. Another way of thinking about the latter condition is that it reflects the constraint that risk preferences can either reduce selection of both contracts or shift preference toward one of the contracts but not alter the budget constraint limiting how many contracts individual traders can purchase.

In Equation (23) the numerator describes the part of the support of probabilities associated with choosing contract C_1. This result will be employed in proving the following proposition, which is the generalization of the results of the previous section.

Proposition 5. *Consider potential traders who are homogeneous in their preferences and have increasing preferences, and consider a market with a uniform distribution of beliefs across these potential traders. Then a necessary condition for a binary prediction market to lead to equilibrium prices outside the convex hull of traders' beliefs is that the market risk premia of the homogeneous market for the two contracts must be of different signs.*

Proof. Without loss of generality, we identify conditions for prices outside the convex hull of beliefs such that $c_1 > p_{1,\max}$ and $c_2 < p_{2,\min}$. Solving Equation (23) for c_1 yields the following condition for prices outside the convex hull:

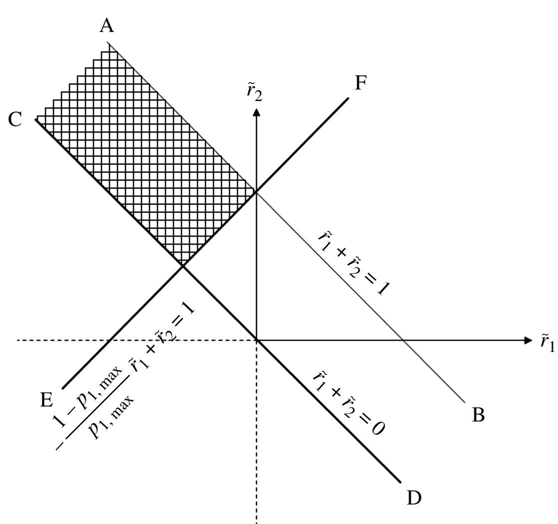
$$c_1 - p_{1,\max} = \frac{-r_1 - (2W_p - (r_1 + r_2))p_{1,\max}}{1 + 2W_p - (r_1 + r_2)} > 0. \quad (26)$$

The denominator is positive, from the constraints on $r_1 + r_2$, and hence from the numerator, the above condition can be written as a linear inequality constraint involving r_1 and r_2:

$$\frac{r_2}{2W_p} > 1 + \frac{1 - p_{1,\max}}{p_{1,\max}} \frac{r_1}{2W_p}. \quad (27)$$

Figure 3 plots the constraints on $\tilde{r}_1$ and $\tilde{r}_2$, where the tilde denotes the normalized market risk premia $r_1/(2W_p)$ and $r_2/(2W_p)$. Admissible values of $\tilde{r}_1$ and $\tilde{r}_2$ are constrained by $0 \le \tilde{r}_1 + \tilde{r}_2 < 1$, which requires that these risk premia lie between lines AB and CD. On line CD, where $\tilde{r}_1 + \tilde{r}_2 = 0$, all traders select one or other contract. One line AB, where $\tilde{r}_1 + \tilde{r}_2 = 1$, no contract is chosen. Therefore admissible values can come arbitrary close to AB but cannot lie on this line. The condition for prices outside the convex hull, in Equation (27), corresponds to points to the northwest of line EF in the figure. The region satisfying these three conditions is the hatched region, over which $r_1 < 0$ and $r_2 > 0$. This proves the proposition.

Figure 3 **Constraints on $\tilde{r}_1$ and $\tilde{r}_2$, and Necessary Condition (27), for Equilibrium Prices Outside the Convex Hull of Probability Beliefs Such That $c_1 > p_{1,\max}$ and $c_2 < p_{2,\min}$**



Notes. Admissible values of risk premia correspond to $0 \le \tilde{r}_1 + \tilde{r}_2 < 1$, which lies between the lines CD and AB (but not including AB). Nonconvex market equilibria with $c_1 > p_{1,\max}$ requires values to the northwest of line EF. The effect of these conditions is that nonconvex prices would occur if $\tilde{r}_1$ and $\tilde{r}_2$ are in the hatched region indicated in the figure.

Proposition 6 (Corollary). *Consider potential traders that are homogeneous in their preferences and have increasing preferences, and consider a market with a uniform distribution of beliefs across these potential traders. The necessary and sufficient conditions for a binary prediction market to lead to equilibrium prices outside the convex hull of traders' beliefs depend only on the ratios r_1/W_p and r_2/W_p.*

Proof. The region in the space of r_1 and r_2 over which nonconvex prices result is defined by the inequalities in Equations (24), (25), and (27). The terms involving r_1 and r_2 and W_p in these equations can be rewritten in terms of $\tilde{r}_1$ and $\tilde{r}_2$ as defined above (see Figure 3), thus proving the proposition.

Proposition 7 (Loss Aversion). *Consider potential traders who are homogeneous in their preferences and have increasing preferences, and consider a market with a uniform distribution of beliefs across these potential traders. Furthermore, all traders have CPT preferences with probability weighting function equal to the probability and no diminishing sensitivity of the value function. Then only the presence of loss aversion in CPT cannot lead to nonconvex prices.*

Proof. The first thing to note about loss aversion is that it might lead some traders to not participate in the market. With the effects of loss aversion alone, i.e., by setting $w_i(p) = p$ and $a_i = 1$ in Equations (8) and (9), we obtain that $p_{1,t} = \Gamma c_1/(1 + (\Gamma - 1)c_1)$ and $p'_{1,t} = c_1/(c_1 + \Gamma(1 - c_1))$. From this, we can calculate market risk premia r_1 and r_2 for the homogeneous market from their definitions (Lemma 2), and using the fact that $p'_{1,t} = 1 - p_{2,t}$, this yields $r_1 = (\Gamma - 1)c_1(1 - c_1)/(1 + (\Gamma - 1)c_1)$ and $r_2 = (\Gamma - 1)c_1(1 - c_1)/(c_1 + \Gamma(1 - c_1))$. With $\Gamma > 1$, both r_1 and r_2 are strictly positive, and if $\Gamma = 1$, they are equal to 0. Because $\Gamma \ge 1$ in the presence of loss aversion, both r_1 and r_2 are nonnegative. Therefore market risk premia do not satisfy the necessary conditions (in Proposition 5) for nonconvex prices, thus proving this proposition.

6.1. Applications

Proposition 5 shows that nonconvex prices require that the market risk premia have opposite signs for the two complementary contracts. In such a market, the contract with negative risk premium is favored (in the above proof, such a contract has been called C_1) and its complement is disfavored compared with a hypothetical case in which traders are all risk neutral.

In the model of §5.2, where probability weights depart from probabilities, the market risk premium for contract j is $md(2c_j - 1)/(1-m)$. If there are no arbitrage openings so that these contract prices sum to 1, then $r_1 + r_2 = md(2c_1 + 2c_2 - 2)/(1-m) = 0$; i.e., the two risk premia sum to zero and together lie on line CD in Figure 3. The sign of the risk premium depends on the value of c_j; i.e., for the event with small mean probability with $c_j < 0.5$, the premium r_j is negative, whereas for the complementary event, it is positive. Of course, this requires $m > 0$ following the model of §5.2; if, instead, $m = 0$, the market risk premium would be 0 for both contracts.

In the model with diminishing sensitivity as described in §5.1, the market risk premium for contract j is $1/(1 + (1/c_j - 1)^a) - c_j$. In a market with no arbitrage openings, it can be shown that $r_1 + r_2 = 0$. Furthermore, for diminishing sensitivity with $a < 1$, the market risk premium is negative for the contract with $c_j > 0.5$ and positive for the other contract.

7. Conclusions

Key results are presented in §6. Market risk premia of the homogeneous market, introduced in Lemma 2, offer a straightforward approach to quantify and understand consequences of different types of preferences for equilibrium prices under the restriction that potential traders all have homogeneous preferences. Proposition 5 discusses the main result—namely, that risk premia of opposite signs are necessary for prices outside the convex hull of probability beliefs. This is seen as the hatched region in Figure 3. This region is the intersection of the admissibility conditions on risk premia, so that markets can function, and the condition for nonconvex prices in Equation (27). The admissibility conditions arise from the following requirements: First, a functioning market in which the market price is well defined is one in which at least some traders choose to hold one of the contracts. Second, risk preferences cannot alter constraints on how many contracts traders can purchase.

When prices occur outside the convex hull of probability beliefs, that contract with a negative market risk premium has an equilibrium price that is larger than the maximum probability for the underlying event. Equivalently, the contract with a positive market risk premium has a price that is lower than the minimum probability for the corresponding event. Risk premia of opposite signs are not sufficient for nonconvex prices, as the hatched region of Figure 3 occupies only a part of its quadrant.

Market risk premia depend only on preferences and equilibrium prices, and not on the support of probability beliefs. Therefore the fact that the conditions for nonconvex prices depend only on the ratios $\tilde{r}_1 = r_1/W_p$ and $\tilde{r}_2 = r_2/W_p$ (Proposition 6) implies that if the support of beliefs is narrow, nonconvex equilibrium prices can generally occur more easily.

Furthermore, for the event associated with the contract having negative market risk premium, nonconvex equilibrium prices can occur more easily when the maximum probability among potential traders is small. This is seen in Figure 3, where in this case, by construction, the negative risk premium (i.e., $\tilde{r}_1$) is associated with contract C_1 and the positive risk premium (i.e., $\tilde{r}_2$) is associated with contract C_2 in the hatched region where nonconvex prices occur. Then in this case, if $p_{1,\max}$ is lower, the slope of line EF is larger. Line EF is constrained to intercept $\tilde{r}_2 = 1$ at $\tilde{r}_1 = 0$, so a lower $p_{1,\max}$ pivots this line with this fixed intercept to increase its slope. This would increase the area of the hatched region in the figure defining where nonconvex prices do occur. Thus nonconvex prices can occur more easily if preferences favor the event with the low mean probability. This is the case of probability weighting in the model of CPT but contrary to that of diminishing sensitivity of the value function.

In the absence of loss aversion, the market risk premia are constrained to lie on line CD in Figure 3, i.e., $\tilde{r}_1 + \tilde{r}_2 = 0$, and in this case all potential traders would enter the market and choose to hold one or other contract. That is not the case with strict loss aversion, in which $\tilde{r}_1 + \tilde{r}_2 > 0$. Here, some potential traders would choose to not enter the market.

With CPT preferences in which only the effect of loss aversion is present, market risk premia for both contracts are strictly positive. Hence nonconvex prices cannot occur with this effect alone (Proposition 7).

7.1. Discussion

Prediction market forecasts can lead to incoherence in decisions involving comparison with risks that are not also priced by the market, if market equilibrium price is outside the convex hull of probability beliefs. This is because nonconvex prices, when interpreted as market probabilities for uncertain events, can induce stochastically dominated choices by the group. Although illustrated in this paper using the case of cumulative prospect theory, the larger lesson is that when features of individual preferences lead to opposite signs of the market risk premia for complementary contracts, nonconvex prices can result. Furthermore, nonconvex prices can occur more easily where the support of probability beliefs is narrower.

Opposite signs of market risk premia for complementary contracts can result through nonlinear probability weights in the model of cumulative prospect theory, or through diminishing sensitivity of the prospect theory value function to deviations from the reference point, or if the utility function has an inflexion point. Nonconvex equilibrium prices result from interactions between preferences and probabilities, and they cannot occur when market risk premia have the same sign for both complementary contracts. For example, with loss aversion of the value function in prospect theory, both risk premia are positive, so equilibrium prices occur within the convex hull of probability beliefs.

Furthermore, with a local change in sign of the utility function's curvature, it would be possible for prices to be outside the convex hull of probability beliefs, analogous to effects of diminishing sensitivity of the prospect theory value function. In prospect theory, different attitudes toward risk when faced with potential gains and losses are an effect of diminishing sensitivity of the value function. This is motivated in part by diminishing perceptions of increasing deviations from the reference point (Kahneman and Tversky 1979). However, the suggestion that people have different attitudes toward risk when faced with potential gains and losses appears earlier in utility theory (Markowitz 1952).

Therefore, the possibility of nonconvex prices needs to be considered in decisions based on market forecasts of event probabilities. When preferences of market traders are unknown, and the possibility that market risk premia for complementary contracts have opposite signs cannot be precluded, the merits of employing markets to make probability forecasts (Arrow et al. 2008) might be weighed against the possibility of inducing group decisions that would be stochastically dominated. Therefore market forecasts should be used with caution, particularly when the forecasts are being applied to decisions that involve comparison with quantities that are not priced by the market.

The origin of nonconvex prices and stochastic dominance in this work is quite different from the origin of stochastic dominance in individual decisions in Prospect theory as presented by Kahneman and Tversky (1979). There, stochastic dominance required nonlinearity of the probability weighting function. Here, it is present even with a linear approximation to the probability weighting function or with diminishing sensitivity to departures from the reference point. This is because, fundamentally, opposite signs of risk premia for complementary contracts are of different origin and reflect overall effects of risk preferences rather than specific forms of the weighting function.

The work presented here is based on several restrictions: homogeneous preferences, two complementary events priced by the market, and an exogenous reference point. It is likely that the results carry over to the more general cases of heterogeneous preferences and markets involving contracts that are defined on a number of mutually exclusive and exhaustive events. In a prediction market setting where traders repeatedly update their choices in response to new information, it is possible that reference points are endogenous (Shalev 2000). Although it is not possible to ascertain the effects of endogenous reference points on market equilibria without constructing an explicit model, the generalization presented in §6 would apply to interpreting equilibria in such a case.

Acknowledgments
This work has been supported by the Climate Decision Making Center, created through a cooperative agreement between the National Science Foundation and Carnegie Mellon University [SES-0345798]; and the Center for

Climate and Energy Decision Making, created through a cooperative agreement between the National Science Foundation and Carnegie Mellon University [SES-0940710]. The author thanks Russell Golman and Teddy Seidenfeld for helpful discussion and two anonymous referees for their suggestions.

References

Andrikogiannopoulou A (2011) Estimating risk preferences from a large panel of real-world betting choices. Working paper, Princeton University, Princeton, NJ.

Arrow KJ, Forsythe R, Gorham M, Hahn R, Hanson R, Ledyard JO, Levmore S, et al. (2008) Economics: The promise of prediction markets. *Science* 320(5878):877–878.

de Finetti B, Machi A, Smith A (1990) *Theory of Probability: A Critical Introductory Treatment*, Vol. 1 (Wiley Interscience, New York).

Galanter E, Pliner P (1974) Cross-modality matching of money against other continua. *Sensation and Measurement* (Reidel, Dordrecht, The Netherlands), 65–76.

Gjerstad S (2005) Risk aversion, beliefs, and prediction market equilibrium. Working paper, University of Arizona, Tucson.

Gollier C (2001) *The Economics of Risk and Time* (MIT Press, Cambridge, MA).

Hansen R (1999) Decision markets. *IEEE Intelligent Systems* 14(3): 16–19.

Helson H (1964) *Adaptation-Level Theory* (Harper & Row, New York).

Kahneman D, Tversky A (1979) Prospect theory: An analysis of decision under risk. *Econometrica* 47(2):263–292.

Manski CF (2006) Interpreting the predictions of prediction markets. *Econom. Lett.* 91(3):425–429.

Markowitz H (1952) The utility of wealth. *J. Political Econom.* 60(2):151–158.

Prelec D (1998) The probability weighting function. *Econometrica* 66(3):497–527.

Rabin M, Thaler RH (2001) Anomalies: Risk aversion. *J. Econom. Perspect.* 15(1):219–232.

Shalev J (2000) Loss aversion equilibrium. *Internat. J. Game Theory* 29(2):269–287.

Snowberg E, Wolfers J (2010) Explaining the favorite-longshot bias: Is it risk-love or misperceptions? *J. Political Econom.* 118(4): 723–746.

Tversky A, Kahneman D (1992) Advances in prospect theory: Cumulative representation of uncertainty. *J. Risk Uncertainty* 5(4):297–323.

Wolfers J, Zitzewitz E (2004) Prediction markets. *J. Econom. Perspect.* 18(2):107–126.

Ashwin K. Seshadri is a senior project scientist at Divecha Centre for Climate Change, Centre for Atmospheric and Oceanic Sciences, Indian Institute of Science, where he studies problems in climate dynamics. He holds an interdisciplinary Ph.D. from the Department of Engineering and Public Policy at Carnegie Mellon University.